ISSN Online: 2756-3898, ISSN Print: 2714-500X https://doi.org/10.47514/phyaccess.2025.5.2.007

Relativistic Study of Motion of Photons in an Elliptical Gravitational Field

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Received 04-09-2025
Accepted for publication 05-11-2025
Published 07-11-2025

Abstract

Fields of elliptical objects can be considered elliptically symmetric, such that the structure, matter, and field of such an object can be defined in terms of an elliptically symmetric space metric. In this study, the need to employ a line element in the gravitational field due to a static and ellipsoidal isolated gravitating mass point was adopted. The relativistic equation of motion of an ellipsoidal star is obtained by applying conditions, such as the covariant and contravariant components of the metric tensor on the metric of the equation, where the metric tensor, affine connections, and geodesic equations were put to work. The contra-variant metric tensor of the basic metric tensor in its covariant form was obtained using the quotient relation. These results offer refined insight into the interpretation of astrophysical observations near non-spherical massive bodies such as oblate stars, elliptical galaxies, or deformed compact objects. The findings contribute to the broader effort of modeling light propagation in realistic, anisotropic gravitational environments within the framework of general relativity.

Keywords: General Relativity; Photons; Elliptical Orbit; Metric Tensor.

I. Introduction

The motion of massive test particles in a classically curved gravitational spacetime has been a subject of significant interest since the early days of general relativity. Such interest is motivated by a wealth of physical applications in astrophysics and cosmology, besides intrinsic intellectual curiosity, as well as by practical issues, such as satellite navigation in space and the definition of relativistic GPS systems [1-2]. The trajectory of a test particle within a given gravitational field in the absence of external forces is known to follow the geodesics of the chosen metric [3].

Light is composed of miniscule corpuscles (Newton), photons (Lewis), or quanta (Einstein) of energy that propagate forward at rapid intrinsic speed c (Maxwell, Hertz, Michelson) with perpendicular electric and magnetic field sets that that self-induce and self-annihilate each other according to laws of induction that trace patterns of waves (Young) through space [4]. The energy that a given photon of light contains is given by $E = hf = {hc}/{\lambda}$ where h is Planck's constant, f is the frequency, and λ, the wavelength of light. Light has no mass [5].

The idea of photon regions, transversely trapping surfaces, in spacetimes without spherical symmetry, characterizes photon regions (some equivalent of photon spheres) "thickened" by oblateness, indicating that null geodesic equations are non-separable. The null geodesic equations show what kind of photon orbit structures emerge in static but non-spherical fields, which are useful in elliptical gravitational field analogues [6]. Therefore, extra terms (linear in r, cosmological constant) affect null geodesics, deflection angles due to gravitational deflection, and light sensing, which helps in comparing how non-standard/ non-spherical and non-asymptotically flat contributions affect photon paths [7]. Looking at radial and spacelike null geodesics between “static sphere observers”, analyze causal structure, and how geodesics behave in relation to the horizons. While still spherical, the addition of cosmological horizon and static spheres allows exploring “non-flat background distortions”; they affirmed that it helps build intuition for how background gravitational fields (not just the central mass) affect photon paths [8]. Furthermore, the analytic expressions for circular photon orbits, energy/angular momentum, as well as epicyclic frequencies, help establish baseline behavior [9].

Fully elliptically distorted gravitational fields (mass distributions with ellipticity rather than just axisymmetric multipoles) with non-axisymmetric perturbations; photon geodesics in such fields are less often treated in detail. Therefore, the deep understanding of the motion of photons in an elliptical gravitational field needs to be explored since the Earth, though approximately spherical, is often modeled as an ellipsoid. The General Theory of Relativity is regarded as the most accepted modern theory of gravitation introduced by Einstein in 1916. In this article, the motion of photons within elliptical gravitational fields is studied in the framework of the General Theory of Relativity.

II. Metric Tensor of an Elliptical Gravitational Mass Point

Fields of elliptical stars or objects can be considered elliptically symmetric, such that the structure, matter, and field of such an object can be defined in terms of an elliptically symmetric space metric.

The metric of a gravitating mass point in its complete form in elliptical polar coordinates is given by (1) [10].

$$ds^2 = \left(1-\frac{2m}{r}\right)c^2dt^2 - \left(1-\frac{2m}{r}\right)^{-1}\left(\frac{r^2+a^2cos^2\theta}{r^2+a^2}\right)dr^2 - (r^2+a^2cos^2\theta)d\theta^2 - (r^2+a^2)sin^2\theta d\phi^2 \quad (1)$$

Where ‘a is a constant in the xy-surface. Thus, it can be said that the meaning of m is as defined in the Schwarzschild equation.

Putting $m = \frac{GM}{c^2}$, it can be seen that for a large value of r, $g \approx \frac{GM}{r^2}$ for M to be the mass of the central body. Then $c = G = 1$ from relativistic units can warrant us to simply have $m = M$ which is dimensionally correct [1]. When $r < a$, the metric is the internal metric of the elliptical field with an incompressible liquid. At the surface $r = a$, the metric coincides with that of a mass point and the outer metric.

Schwarzschild’s metric of spherical symmetry is given in (2).

$$ds^2 = c^2d\tau^2 = c^2\left(1-\frac{2GM}{c^2r}\right)dt^2 - \left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \quad (2)$$

This has been the basis of the theoretical investigation of gravitational phenomena in Einstein’s theory of General Relativity [2]. Equation (2) has the exact solution of Einstein’s field equation given by (3).

$$G_{\alpha\beta} = \frac{-8\pi G}{c^4}T_{\alpha\beta} \quad (3)$$

Where $G_{\alpha\beta}$ is the Einstein Tensor, G is the gravitational constant, c is the speed of light, and $T_{\alpha\beta}$ is the stress tensor.

From (1), (4) can be obtained using the Riemann geometry.

$$\left.\begin{array}{c} g_{00} = \left(1-\frac{2m}{r}\right) \\ g_{11} = -\left(1-\frac{2m}{r}\right)^{-1}\left(\frac{r^2+a^2cos^2\theta}{r^2+a^2}\right) \\ g_{33} = -(r^2+a^2)sin^2\theta \\ g_{\beta\gamma} = 0, \text{for } \beta \neq \gamma \end{array}\right\} \quad (4)$$

Deducing the contra-variant metric tensor using the quotient theorem of tensor analysis given as (5).

$$g^{1\beta} = \frac{1}{g_{\beta 1}} \quad (5)$$

By applying the coefficient of the affine connection, which is defined as the sum of the product of contravariant and covariant metric tensor [2], given as (6).

$$\Gamma^{\sigma}_{\alpha\lambda} = {}^1/_2\, g^{\sigma\beta}\left(g_{\alpha\beta,\lambda} + g_{\beta\lambda,\alpha} - g_{\lambda\alpha,\beta}\right) \quad (6)$$

where $g_{\alpha\beta,\lambda} = \frac{dg_{\alpha\beta}}{dx^\lambda}$ such that $\lambda = 0,1,2,3$.

Using (6), the following equation can be obtained.

$$\Gamma^0_{01} = \frac{\rho'}{2} \quad (7)$$

$$\Gamma^0_{02} = \frac{\dot{\rho}}{2} \quad (8)$$

$$\Gamma^1_{00} = \frac{\rho' e^{\rho}}{2e^{\sigma}}\left(\frac{r^2+a^2}{r^2+a^2cos^2\theta}\right) \quad (9)$$

$$\Gamma^1_{11} = \frac{\sigma'}{2} + \frac{r}{r^2+a^2cos^2\theta} - \frac{r}{r^2+a^2} \quad (10)$$

$$\Gamma^1_{12} = \frac{\dot{\sigma}}{2} - \frac{a^2\sin 2\theta}{2} \quad (11)$$

$$\Gamma^1_{22} = \left(\frac{-r(r^2+a^2)}{e^{\sigma}(r^2+a^2cos^2\theta)}\right) \quad (12)$$

$$\Gamma^1_{33} = \frac{-r(r^2+a^2)sin^2\theta}{e^{\sigma}(r^2+a^2cos^2\theta)} \quad (13)$$

$$\Gamma^2_{00} = \frac{\dot{\rho}e^{\rho}}{2(r^2+a^2cos^2\theta)} \quad (14)$$

$$\Gamma^2_{11} = \frac{-\dot{\sigma}e^{\sigma}}{2} + \frac{e^{\sigma}a^2\sin 2\theta}{2(r^2+a^2cos^2\theta)(r^2+a^2)} \quad (15)$$

$$\Gamma^2_{12} = \frac{r}{(r^2+a^2cos^2\theta)} \quad (16)$$

$$\Gamma^2_{22} = \frac{-a^2\sin 2\theta}{(r^2+a^2cos^2\theta)} \quad (17)$$

$$\Gamma^2_{33} = \frac{-(r^2+a^2)\sin 2\theta}{2(r^2+a^2cos^2\theta)} \quad (18)$$

$$\Gamma^3_{13} = \frac{r}{r^2+a^2} \quad (19)$$

$$\Gamma^3_{23} = \cot\theta \quad (20)$$

III. Motion of Comet Particles in the Vicinity of an Ellipsoidal Mass

The general relativistic equation of motion for particles of non-zero rest mass in a gravitational field [2] is given as;

$$\frac{d^2x^\sigma}{d\tau^2} + \Gamma^{\sigma}_{\alpha\lambda}\frac{dx^\alpha}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \quad (21)$$

Where τ is the proper time.

Therefore, the equations of motion are given explicitly as follows;

When $\sigma = 0, x^0 = ct, x^1 = r, x^2 = \theta, x^3 = \phi$

$$\frac{d^2x^0}{d\tau^2} + \Gamma^{0}_{\alpha\lambda}\frac{dx^\alpha}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \quad (22)$$

$\frac{d^2(ct)}{d\tau^2} + \Gamma_{01}^0 \frac{dx^0}{d\tau}\frac{dx^1}{d\tau} + \Gamma_{02}^0 \frac{dx^0}{d\tau}\frac{dx^2}{d\tau} = 0$ (23)

$\frac{d^2(ct)}{d\tau^2} + \Gamma_{01}^0 \frac{d(ct)}{d\tau}\frac{dr}{d\tau} + \Gamma_{02}^0 \frac{d(ct)}{d\tau}\frac{d\theta}{d\tau} = 0$ (24)

$c\ddot{t} + \Gamma_{01}^0 c\dot{t}\dot{r} + \Gamma_{02}^0 c\dot{t}\dot{\theta} = 0$ (25)

Substituting (2) and (3) into (21), yield (26).

$c\ddot{t} + v' c\dot{t}\dot{r} + \dot{v} c\dot{t}\dot{\theta} = 0$ (26)

Equation (21) is the time equation of motion for comets in this gravitational field.

Similarly, setting $\sigma = 1, x^0 = ct, x^1 = r, x^2 = \theta, x^3 = \phi$

$\frac{d^2x^1}{d\tau^2} + \Gamma_{\alpha\lambda}^1 \frac{dx^\alpha}{d\tau}\frac{dx^\lambda}{d\tau} =$ (27)

$\frac{d^2r}{d\tau^2} + \Gamma_{00}^1 \frac{dx^0}{d\tau}\frac{dx^0}{d\tau} + \Gamma_{11}^1 \frac{dx^1}{d\tau}\frac{dx^1}{d\tau} + 2\Gamma_{12}^1 \frac{dx^1}{d\tau}\frac{dx^2}{d\tau} + \Gamma_{22}^1 \frac{dx^2}{d\tau}\frac{dx^2}{d\tau} + \Gamma_{33}^1 \frac{dx^3}{d\tau}\frac{dx^3}{d\tau} = 0$

$\ddot{r} + \Gamma_{00}^1 \dot{t}^2 + \Gamma_{11}^1 \dot{r}^2 + 2\Gamma_{12}^1 \dot{r}\dot{\theta} + \Gamma_{22}^1 \dot{\theta}^2 + \Gamma_{33}^1 \dot{\phi}^2 = 0$ (28)

Substituting (9), (10), (11), (12), and (13) into (28), presents (29).

$\ddot{r} + \frac{\rho' e^\rho}{2e^\sigma}\left(\frac{r^2+a^2}{r^2+a^2cos^2\theta}\right)\dot{t}^2 + \left(\frac{\sigma'}{2} + \frac{r}{r^2+a^2cos^2\theta} - \frac{r}{r^2+a^2}\right)\dot{r}^2 + 2\left(\frac{\dot{\sigma}}{2} - \frac{a^2\sin 2\theta}{2}\right)\dot{r}\dot{\theta} - \frac{r(r^2+a^2)}{e^\sigma(r^2+a^2cos^2\theta)}\dot{\theta}^2 - \frac{r(r^2+a^2)sin^2\theta}{e^\sigma(r^2+a^2cos^2\theta)}\dot{\phi}^2 = 0$ (29)

This is the radial equation of motion of comet particles in this gravitational field.

Also, for $\sigma = 2$, and $\sigma = 3$ in (25), the polar and azimuthal equations of motion in the inner gravitational field of an ellipsoidal star are obtained respectively as;

$\ddot{\theta} + \frac{\dot{\rho}e^\rho}{2(r^2+a^2cos^2\theta)}\dot{t}^2 + \left(\frac{-\dot{\sigma}e^\sigma}{2} + \frac{e^\sigma a^2\sin 2\theta}{2(r^2+a^2cos^2\theta)(r^2+a^2)}\right)\dot{r}^2 + 2\frac{r}{(r^2+a^2cos^2\theta)}\dot{r}\dot{\theta} - \frac{a^2\sin 2\theta}{(r^2+a^2cos^2\theta)}\dot{\theta}^2 - \frac{(r^2+a^2)\sin 2\theta}{2(r^2+a^2cos^2\theta)}\dot{\phi}^2 = 0$ (30)

and

$\ddot{\phi} + 2\frac{r}{r^2+a^2}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0$ (31)

The radial equation (29) is space-time related, which is similar to that of the Schwarzschild radial equation of motion, with only the inclusion of adjustable parameters such as $(a\cos\theta)^2$.

Note that when $\sigma = 0$, the geodesic equation of motion is explicitly radial, polar, and time coordinate derivative of proper time, which is different from the Schwarzschild. Geodesic equation $\sigma = 3$ is purely space-related, while the rest are space-time related.

Now, considering Photons moving in the equatorial plane $(\theta = \pi/2)$ of the ellipsoidal star. Thus (1) reduces to:

$c^2 = \left(1 - \frac{2m}{r}\right)c^2\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{r^2}{r^2+a^2}\right)\dot{r}^2 - -(r^2 + a^2)\dot{\phi}^2$ (32)

Explicit expressions for $\dot{t}$ and $\dot{\phi}$ are obtained respectively by solving the geodesic equations (26) and (31).

From time like geodesic equation, at the equatorial, (26) becomes,

$\frac{\ddot{t}}{\dot{t}} + v'\dot{r} = 0$ (33)

Taking the integral of (33) with respect to t, gives (34).

$\dot{t} = e^{A-v}$ (34)

where A is an arbitrary constant

Assuming that $e^A = K$ and $e^v = \left(1 - \frac{2m}{r}\right)$, and substituting into (34), (34) becomes,

$\dot{t} = K\left(1 - \frac{2m}{r}\right)^{-1}$ (35)

Considering the geodesic equation, given by (31) at the equatorial plane, and dividing through by $\dot{\Phi}$, then (35) becomes,

$\frac{\ddot{\phi}}{\dot{\phi}} + 2\frac{r}{r^2+a^2}\dot{r} = 0$ (36)

Taking the integral of (36) with respect to τ yields the following:

$\int \frac{\ddot{\phi}}{\dot{\phi}} d\tau + \int 2\frac{r}{r^2+a^2}\dot{r}d\tau = 0$ (37)

$\int \frac{\ddot{\phi}}{\dot{\phi}} d\tau + \int 2\frac{r}{r^2+a^2}dr = 0$ (38)

$ln\dot{\phi} + \ln(r^2+a^2) = \ln B$ (39)

$\dot{\phi}(r^2+a^2) = B$

$\therefore\ \dot{\phi} = \frac{B}{(r^2+a^2)}$ (40)

Where B is also an arbitrary constant.

IV. Motion of Photons in the Vicinity of an Ellipsoidal Isolated Mass

Based on the General Theory of Relativity, light (photon) moves along a curve that has zero interval (null geodesic), i.e.

$ds^2 = c^2d\tau^2 = 0$ (41)

Equation (40) implies that light is timeless.

Considering a photon moving in the equatorial plane, then $\theta = \frac{\pi}{2}, d\theta = 0,, \ sin\theta = 1$, and $cos\theta = 0$. Equations (1) and (41) yield (42).

$0 = \left(1 - \frac{2m}{r}\right)c^2dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{r^2}{r^2+a^2}\right)dr^2 - (r^2 + a^2)d\phi^2$ (42)

Now, describing the motion of a photon with respect to a parameter w and then dividing through by dw^2, gives (43).

$0 = \left(1 - \frac{2m}{r}\right)c^2\frac{dt^2}{dw^2} - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{r^2}{r^2+a^2}\right)\frac{dr^2}{dw^2} - (r^2 + a^2)\frac{d\phi^2}{dw^2}$ (43)

Assuming $\dot{t} = \frac{dt}{dw}$, $\dot{r} = \frac{dr}{dw}$, and $\dot{\phi} = \frac{d\phi}{dw}$, then (44) can be obtained.

$0 = \left(1 - \frac{2m}{r}\right)c^2\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{r^2}{r^2+a^2}\right)\dot{r}^2 - (r^2 + a^2)\dot{\phi}^2$ (44)

Expressing r as a function of ϕ i.e $r = r(\phi)$, then $\dot{r} = \dot{\phi}\, dr/d\phi$ and substituting (31) and (40) into (45) yields

$\frac{B^2r^2}{(r^2+a^2)^3}\left(\frac{dr}{d\phi}\right)^2 = c^2K^2 - \frac{B^2}{(r^2+a^2)}\left(1 - \frac{2m}{r}\right)$ (45)

Letting $r(\phi) = 1/u(\phi)$ such that $\frac{dr}{d\phi} = \frac{dr}{du}\frac{du}{d\phi} = -u^{-2}\frac{du}{d\phi}$ and multiplying through by $\left(\frac{B^2}{(1+a^2u^2)}\right)^{-1}$ in (40), (45) can take the form given in (46).

$\left(\frac{du}{d\phi}\right)^2 = \frac{c^2K^2(1+a^2u^2)^3}{B^2} - u^2(1+a^2u^2)^2 + 2mu^3(1 + a^2u^2)^2$ (46)

Differentiating (46) with respect to u and collecting like terms yields (47).

$$\frac{d^2u}{d\phi^2} = u\left[\frac{3a^2c^2K^2}{B^2} + 3mu - 1\right](1 + a^2u^2)^2 - 2u^3a^2[1 + 2mu](1 + a^2u^2) \quad (47)$$

Thus, (47) is the relativistic equation of motion of a photon in the vicinity of an ellipsoidal mass.

V. Conclusion

The equation of motion of a photon is obtained from the space metric by using the condition that light travels along a null geodesic. The equation of a photon in the equatorial plane of an ellipsoidal plane differs from the equation of motion of a photon obtained from Schwarzschild's metric. This indicates that at elliptical gravity, the motion is unique. The relativistic equation of motion of a photon in the vicinity of an ellipsoidal mass is useful for both spherical and non-spherical objects. These results offer refined insight into the interpretation of astrophysical observations near non-spherical massive bodies such as oblate stars, elliptical galaxies, or deformed compact objects. The findings contribute to the broader effort of modeling light propagation in realistic, anisotropic gravitational environments within the framework of general relativity.

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